Arbitrage of the first kind and filtration enlargements

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(based on a joint work with C. Fontana and C. Kardaras)

Outline of the talk

- Problem formulation and motivation
- Progressive enlargement of filtration
- Initial enlargement of filtration
- Examples

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The problem

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Mathematically:

- ▷ market : $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, S)$, with $\mathbb{F}$ satisfying the usual conditions, $S = (S^i)_{i=1,\dots,d}$ non-negative semimartingale, $S^0 \equiv 1$.
- ▷ additional information:
  - progressive enlargement of filtration (with any random time)
  - initial enlargement of filtration
- ▷ arbitrage profits: ...(some motivation first)...

The basic example

- ▷ Let W be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}^W, \mathbb{P})$.
- ▷ Let S represent the discounted price of an asset and be given by

$$S_t = \exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right), \qquad \sigma > 0 \text{ given.}$$

- ▷ Let $S_t^* := \sup\{S_u,\ u \leq t\}$ and define the random time

$$\boldsymbol{\tau} := \sup\{t \ :\ S_t = S_\infty^*\} = \sup\{t \ :\ S_t = S_t^*\}$$

- ▷ An agent with information τ can follow the **arbitrage strategy**

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- An agent with information τ can follow the **arbitrage strategy**

"buy at $t = 0$ and sell at $t = \tau$"

Remark. Here τ is an **honest time**: $\forall \ t \ge 0 \ \exists \ \xi_t \ \mathcal{F}_t^W$-measurable s.t. $\tau = \xi_t$ on $\{\tau \le t\}$ (e.g., $\xi_t := \sup\{u \le t : S_u = \sup_{r \le t} S_r\}$).

Different notions of arbitrage

$\mathcal{X}(\mathbb{F}, S)$ admissible wealth processes: $X^{x,H} := x + \int_0^{\cdot} H_t \mathrm{d}S_t \ \geq 0$

We use the notation:

- NA($\mathbb{F}, S$): there is no $X^{1,H} \in \mathcal{X}(\mathbb{F}, S)$ s.t. $\mathbb{P}\left[X_\infty^{1,H} \geq 1\right] = 1$, $\mathbb{P}\left[X_\infty^{1,H} > 1\right] > 0$.
- NFLVR($\mathbb{F}, S$): there are no $\epsilon > 0$, $0 \leq \delta_n \uparrow 1$, $X^{1,H^n} \in \mathcal{X}(\mathbb{F}, S)$ s.t. $\mathbb{P}\left[X_\infty^{1,H^n} > \delta_n\right] = 1$, $\mathbb{P}\left[X_\infty^{1,H^n} > 1 + \epsilon\right] \geq \epsilon$.
- NA1($\mathbb{F}, S$): there is no $\xi \geq 0$ with $\mathbb{P}\left[\xi > 0\right] > 0$ s.t. for all $x > 0$, $\exists X \in \mathcal{X}(\mathbb{F}, S)$ with $X_0 = x$ and $\mathbb{P}\left[X_\infty \geq \xi\right] = 1$.

Remark. NA1 (Kardaras, 2010) $\Longleftrightarrow$ BK (Kabanov, 1997) $\Longleftrightarrow$ NUPBR (Karatzas,Kardaras 2007)

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- NA1 $\iff$ $\exists$ supermartingale deflator (Karatzas,Kardaras'07):
  $Y > 0$, $Y_0 = 1$ s.t. YX is a supermartingale $\forall X \in \mathcal{X}$

  $\iff$ $\exists$ loc. martingale deflator (Takaoka,Schweizer'13, Song'13):
  $Y > 0$, $Y_0 = 1$ s.t. YX is a local martingale $\forall X \in \mathcal{X}$

  $\iff$ $\exists$ treadable loc. martingale deflator (A.F.K.'14):
  Y local martingale deflator s.t. $1/Y \in \mathcal{X}$ (up to $\mathbb{Q} \sim \mathbb{P}$)

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- As seen in the basic example, **NA and NFLVR easily fail under additional information**.
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- NA1 is the minimal condition in order to proceed with utility maximization.
- NA1 is stable under change of numéraire.
- NA1 is equivalent to the existence of a numéraire portfolio X^* (= growth optimal portfolio = log optimal portfolio), in which case $1/X^*$ is a supermartingale deflator.

Some related work (NA1 preservation)

On progressive enlargement:

- Fontana, Jeanblanc, Song 2013:
  S continuous, PRP, τ honest and avoids all $\mathbb{F}$-stopping times, NFLVR in the original market. Then in the enlarged market:
  - on $[0,\infty)$: NA1, NA and NFLVR all fail;
  - on $[0,\tau]$: NA and NFLVR fail, but NA1 holds.
- Kreher 2014:
  all $\mathbb{F}$-martingales are continuous, τ avoids all $\mathbb{F}$-stopping times, NFLVR in the original market.
- Aksamit, Choulli, Deng, Jeanblanc 2013:
  using optional stochastic integral, (S quasi-left-continuous).

On initial enlargement: nothing in the literature that we are aware of. Some work in progress by Jeanblanc et al.

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Progressive enlargement of filtrations

- Let τ be a **random time** (= positive, finite, $\mathcal{F}$-measurable r.v.).

- Consider the **progressively enlarged filtration** $\mathbb{G} = (\mathcal{G}_t)_{t \in \mathbb{R}_+}$,

$$\mathcal{G}_t := \{B \in \mathcal{F} \mid B \cap \{\tau > t\} = B_t \cap \{\tau > t\} \text{ for some } B_t \in \mathcal{F}_t\}.$$

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- Jeulin-Yor theorem ensures that $\mathcal{H}'$-hypothesis holds **up to τ**: every $\mathbb{F}$-semimartingale remains a $\mathbb{G}$-semimartingale up to time τ (in particular S^τ is a $\mathbb{G}$-semimartingale).

Our main tools

- The Azéma supermartingale associated to τ:

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- Let A be the $\mathbb{F}$-dual optional projection of $\mathbb{I}_{[\![\tau,\infty[\![}$, so that $\Delta A_\sigma = \mathbb{P}[\tau = \sigma | \mathcal{F}_\sigma]$ for all $\mathbb{F}$-stopping times σ.

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- Define the stopping time $\zeta := \inf\left\{t \in \mathbb{R}_+ \mid Z_t = 0\right\} \geq \tau$.

- Define $\Lambda := \{\zeta < \infty,\ Z_{\zeta-} > 0,\ \Delta A_\zeta = 0\} \in \mathcal{F}_\zeta$

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$= \text{set where } Z \text{ jumps to zero after } \tau$

- and define

$$\boxed{\eta := \zeta \mathbb{I}_\Lambda + \infty \mathbb{I}_{\Omega \setminus \Lambda}}$$

Note that $\tau < \eta$; $\eta =$ **time when Z jumps to zero after τ**.

Representation pair associated with τ

Theorem (Itô,Watanabe 1965, Kardaras 2014).
The Azéma supermartingale Z admits the following multiplicative decomposition:

$$\boxed{Z = \mathbf{L}(1 - K),}$$

where:

- L is a nonnegative $\mathbb{F}$-local martingale with $L_0 = 1$,
- K is a nondecreasing $\mathbb{F}$-adapted process with $0 \leq K \leq 1$,
- for any nonnegative optional processes V on $(\Omega, \mathbb{F})$,

$$\mathbb{E}[V_\tau] = \mathbb{E}\left[\int_{\mathbb{R}_+} V_t \, L_t \mathrm{d}K_t\right].$$

▷▷ Together with the stopping time $\boldsymbol{\eta}$, the local martingale $\mathbf{L}$ will play a main role in our results.

Back to the basic example

Asset price process: $S_t = \exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right)$

Random time: $\tau := \sup\{t \ : \ S_t = S^*_\infty\}$

In this case

$$Z_t = \mathbb{P}\left[\tau > t \mid \mathcal{F}_t\right] = \frac{S_t}{S^*_t}$$

Therefore:

▷ $\eta = \infty$ and $L = S$

▷ $Y := 1/L^\tau = 1/S^\tau$ is a local martingale deflator for S^τ in $\mathbb{G}$.

$\Rightarrow$ NA1 holds while NA and NFLVR fail.

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$\Rightarrow$ NA1 holds while NA and NFLVR fail.

Remarks.

1) Analogous situation for $\tau' := \sup\{t \ : \ S_t = a\}$, $0 < a < 1$.

2) The decomposition $Z_t = L_t/L^*_t$ holds for a wide class of honest times (see Nikeghbali,Yor 2006, Kardaras 2013, A.,Penner 2014)

Local martingales in the progressively enlarged filtration

Remember: η is the time when Z jumps to zero after τ.

Proposition. Let X be a nonnegative $\mathbb{F}$-local martingale such that $X = 0$ on $[\![\eta, \infty[\![$. Then X^τ / L^τ is a $\mathbb{G}$-local martingale.

- ▷ The main tool in the proof of the proposition is the multiplicative decomposition of Z.

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As an immediate consequence we have the following

Key-Proposition. Suppose there exists a local martingale deflator M for S in $\mathbb{F}$ such that $M = 0$ on $[\![\eta, \infty[\![$. Then M^τ / L^τ is a local martingale deflator for S^τ in $\mathbb{G}$.

An important lemma

$\Longrightarrow$ To have preservation of the NA1 property, given a deflator for S in $\mathbb{F}$, we want to "kill it" from η on.

We will do it with the help of the following lemma.

Lemma. Let D be the $\mathbb{F}$-predictable compensator of $\mathbb{I}_{[\![\eta,\infty[\![}$. Then:

- $\Delta D < 1$ $\mathbb{P}$-a.s. ($\Rightarrow \mathcal{E}(-D) > 0$ and nonincreasing);
- $\mathcal{E}(-D)^{-1}\mathbb{I}_{[\![0,\eta[\![}$ is a local martingale on $(\Omega, \mathbb{F}, \mathbb{P})$.

Main idea: for any predictable time σ on $(\Omega, \mathbb{F})$,

$$\Delta D_\sigma = \mathbb{P}\left[\eta = \sigma \mid \mathcal{F}_{\sigma-}\right] < 1 \text{ on } \{\sigma < \infty\}.$$

NA1 under progressive enlargement: one fixed S

- We have preservation of NA1 under the condition: S does not jump when Z jumps to zero:

  Theorem (one fixed S). Suppose $\mathbb{P}[\eta < \infty, \, \Delta S_\eta \neq 0] = 0$. If NA1$(\mathbb{F}, S)$holds, then NA1$(\mathbb{G}, S^\tau)$holds.

Proof of the theorem

Recall: D is the $\mathbb{F}$-predictable compensator of $\mathbb{I}_{[\![\eta,\infty[\![}$.

- NA1$(\mathbb{F}, S)$ $\Rightarrow$ $\exists \widehat{X} \in \mathcal{X}(\mathbb{F}, S)$ s.t. $Y := (1/\widehat{X})$ is a local martingale deflator for S in $\mathbb{F}$ ($\Rightarrow \Delta Y = 0$ when $\Delta S = 0$).
- In order to apply the Key-Proposition, we need a deflator for S in $\mathbb{F}$ that vanishes on the set $[\![\eta, \infty[\![$.

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Recall: D is the $\mathbb{F}$-predictable compensator of $\mathbb{I}_{[\![\eta,\infty[\![}$.

- $\text{NA1}(\mathbb{F}, S) \Rightarrow \exists \widehat{X} \in \mathcal{X}(\mathbb{F}, S)$ s.t. $Y := (1/\widehat{X})$ is a local martingale deflator for S in $\mathbb{F}$ ($\Rightarrow \Delta Y = 0$ when $\Delta S = 0$).
- In order to apply the Key-Proposition, we need a deflator for S in $\mathbb{F}$ that vanishes on the set $[\![\eta, \infty[\![$.
- Let $M := Y\mathcal{E}(-D)^{-1}\mathbb{I}_{[\![0,\eta[\![}$ ($\Rightarrow \{M > 0\} = [\![0, \eta]\!]$).
- By the Lemma, $MS - \left[\mathcal{E}(-D)^{-1}\mathbb{I}_{[\![0,\eta[\![}, YS\right]$ $\mathbb{F}$-local martingale.

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- By the Lemma, $MS - \left[\mathcal{E}(-D)^{-1}\mathbb{I}_{[\![0,\eta[\![}, YS\right]$ $\mathbb{F}$-local martingale.
- We want M to be a deflator for S in $\mathbb{F}$, so we need to show that the quadratic covariation part is an $\mathbb{F}$-local martingale.
- $\Delta S_\eta = 0 \Rightarrow \Delta(YS)_\eta = 0 \Rightarrow [..,..] = \left[\mathcal{E}(-D)^{-1}, YS\right]$, which is indeed an $\mathbb{F}$-local martingale.

$\Rightarrow$ $(\hat{X}^{-1}\mathcal{E}(-D)^{-1}L^{-1})^\tau$ is a deflator for S^τ in $\mathbb{G}$

NA1 under progressive enlargement: any S

Theorem (general stability). TFAE:

1) for any S s.t. NA1($\mathbb{F}, S$) holds, NA1($\mathbb{G}, S^\tau$)holds;
2) $\eta = \infty$ $\mathbb{P}$-a.s.;
3) For every nonnegative local martingale X on $(\Omega, \mathbb{F}, \mathbb{P})$, the process X^τ / L^τ is a local martingale on $(\Omega, \mathbb{G}, \mathbb{P})$;
4) The process $1/L^\tau$ is a local martingale on $(\Omega, \mathbb{G}, \mathbb{P})$.

Proof of the theorem

2) $\Rightarrow$ 1): from previous Theorem.

1) $\Rightarrow$ 2): suppose $\mathbb{P}\left[\eta < \infty\right] > 0$. Define

$$S := \mathcal{E}(-D)^{-1}\mathbb{I}_{[\![0,\eta[\![}.$$

Then S is a $\mathbb{F}$-local martingale, and S^τ is nondecreasing with $\mathbb{P}\left[S_\tau > 1\right] > 0$. Hence NA1$(\mathbb{F}, S)$holds, but NA1$(\mathbb{G}, S^\tau)$fails.

2) $\Rightarrow$ 3): from the Proposition.

3) $\Rightarrow$ 4): trivial.

4) $\Rightarrow$ 2): uses properties of the processes L and K appearing in the multiplicative decomposition of Z.

Proposition. Let X be a nonnegative $\mathbb{F}$-supermartingale. Then, the process X^τ/L^τ is a $\mathbb{G}$-supermartingale.

Remark. This can be used to establish that for any semimartingale X on $(\Omega, \mathbb{F}, \mathbb{P})$, the process X^τ is a semimartingale on $(\Omega, \mathbb{G}, \mathbb{P})$.

Indeed:

- By the Proposition, $\forall$ X nonnegative bounded $\mathbb{F}$-local martingale $\Rightarrow$ X^τ/L^τ and $1/L^\tau$ are $\mathbb{G}$-semimartingales $\Rightarrow$ X^τ is a $\mathbb{G}$-semimartingale.
- From the semimartingale decomposition + localisation, same result for any $\mathbb{F}$-semimartingale X.

Initial enlargement of filtrations

- Let **J** be an $\mathcal{F}$-measurable random variable taking values in a Lusin space $(E, \mathcal{B}_E)$, where $\mathcal{B}_E$ denotes the Borel σ-field of E.

- Let $\mathbb{G} = (\mathcal{G}_t)_{t \in \mathbb{R}_+}$ be the right-continuous augmentation of the filtration $\mathbb{G}^0 = (\mathcal{G}_t^0)_{t \in \mathbb{R}_+}$ defined by

$$\mathcal{G}_t^0 := \mathcal{F}_t \vee \sigma(J), \quad t \in \mathbb{R}_+.$$

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- Let $\gamma : \mathcal{B}_E \mapsto [0,1]$ be the law of J ($\gamma[B] = \mathbb{P}[J \in B]$, $B \in \mathcal{B}_E$).

- For all $t \in \mathbb{R}_+$, let $\boldsymbol{\gamma_t} : \Omega \times \mathcal{B}_E \mapsto [0,1]$ be a regular version of the $\mathcal{F}_t$-conditional law of J.

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Jacod's hypothesis. We assume

$$\gamma_t \ll \gamma \quad \mathbb{P}\text{-a.s.}, \quad t \in \mathbb{R}_+.$$

This ensures the $\mathcal{H}'$-hypothesis and that we can apply Stricker& Yor calculus with one parameter ($\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ separable).

Our main tools

$\mathcal{O}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{F})$) is the $\mathbb{F}$-optional (resp. pred.) σ-field on $\Omega \times \mathbb{R}_+$

Lemma. There exists a $\mathcal{B}_E \otimes \mathcal{O}(\mathbb{F})$-measurable function $E \times \Omega \times \mathbb{R}_+ \ni (x, \omega, t) \mapsto p_t^x(\omega) \in [0, \infty)$, càdlàg in $t \in \mathbb{R}_+$ s.t.:

- $\forall t \in \mathbb{R}_+$, $\gamma_t(\mathrm{d}x) = \mathbf{p_t^x}\,\gamma(\mathrm{d}x)$ holds $\mathbb{P}$-a.s;
- $\forall x \in E$, $p^x = (p_t^x)_{t \in \mathbb{R}_+}$ is a martingale on $(\Omega, \mathbb{F}, \mathbb{P})$.

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▷ For every $x \in E$ define

$$\zeta^x := \inf\{t \in \mathbb{R}_+ \mid p_t^x = 0\}.$$

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- ▷ For every $x \in E$ define

$$\zeta^x := \inf\{t \in \mathbb{R}_+ \mid p_t^x = 0\}.$$

- ▷ Let $\Lambda^x := \{\zeta^x < \infty, p^x_{\zeta^x -} > 0\} \in \mathcal{F}_{\zeta^x}$ and define

$$\boxed{\boldsymbol{\eta}^{\mathbf{x}} := \zeta^x \mathbb{I}_{\Lambda^x} + \infty \mathbb{I}_{\Omega \setminus \Lambda^x}, \quad x \in E}$$

Note that $\boldsymbol{\eta}^{\mathbf{x}}$ ($=$ **time at which p^x jumps to zero**) is a stopping time on $(\Omega, \mathbb{F})$.

NA1 under initial enlargement

▷ Similar results for the martingale deflators lead to:

Theorem (one fixed S). Let $\mathbb{P}[\eta^x < \infty, \Delta S_{\eta^x} \neq 0] = 0$ γ-a.e. If NA1($\mathbb{F}, S$)holds, then NA1($\mathbb{G}, S$) holds.

Theorem (general stability). TFAE:

1) $\eta^x = \infty$ $\mathbb{P}$-a.s. for γ-a.e $x \in E$.
2) for all $X \geq 0$ $\mathcal{B}_E \otimes \mathcal{O}(\mathbb{F})$-meas. s.t. X^x $\mathbb{F}$-loc.martingale vanishing on $[\![\eta^x, \infty[\![$ γ-a.e., X^J/p^J is a $\mathbb{G}$-loc.martingale
3) The process $1/p^J$ is a $\mathbb{G}$-loc.martingale

And 1) $\Rightarrow$ For any S s.t. NA1($\mathbb{F}, S$)holds, NA1($\mathbb{G}, S$) also holds.

▷ Some care for the converse; we can derive $\mathcal{H}'$-hyp.

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Example 1: progressively enlarged filtration

▷ Consider $\zeta : \Omega \mapsto \mathbb{R}_+$ such that $\mathbb{P}[\zeta > t] = \exp(-t), \forall\ t \in \mathbb{R}_+$.

▷ Let $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ be the smallest filtration that satisfies the usual hypotheses and makes ζ a stopping time.

▷ Define $\boldsymbol{\tau} := \zeta/2$.

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▷ Note that $Z_t := \mathbb{P}[\tau > t | \mathcal{F}_t] = \exp(-t)\mathbb{I}_{\{t<\zeta\}}$ for all $t \in \mathbb{R}_+$.

▷ Note that $\zeta = \inf\{t \geq 0 \mid Z_t = 0\} =: \eta < \infty$ $\mathbb{P}$-a.s.

▷ The $\mathbb{F}$-pred. comp. of $\mathbb{I}_{[\![\eta,\infty[\![}$ is $D := (\eta \wedge t)_{t \in \mathbb{R}_+}$.

Example 1: progressively enlarged filtration

▷ Consider $\zeta : \Omega \mapsto \mathbb{R}_+$ such that $\mathbb{P}[\zeta > t] = \exp(-t), \forall\ t \in \mathbb{R}_+$.

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▷ The $\mathbb{F}$-pred. comp. of $\mathbb{I}_{[\![\eta,\infty[\![}$ is $D := (\eta \wedge t)_{t \in \mathbb{R}_+}$.

▷ **S** $:= \mathcal{E}(-D)^{-1}\mathbb{I}_{[\![0,\eta[\![} = \exp(D)\mathbb{I}_{[\![0,\eta[\![}$, that is, $S_t = \exp(t)\mathbb{I}_{\{t<\zeta\}}$.

▷ S nonnegative $\mathbb{F}$-martingale $\Rightarrow$ NA1($\mathbb{F}, S$).

▷ But S is strictly increasing up to $\tau \Rightarrow$ NA1($\mathbb{G}, S^\tau$) fails.

Example 2: initially enlarged filtration

- Consider a Poisson(λ) process N stopped at time $T \in (0, \infty)$.
- Let $\mathbb{F}$ be the right-cont. filtration generated by N and $\mathbf{J} := N_T$.
- Then (Grorud,Pontier 2001) $p_T^x = \mathrm{e}^{-\lambda T} x!/(\lambda T)^x \mathbb{I}_{\{N_T = x\}}$ and

$$p_t^x = \mathrm{e}^{-\lambda t} \frac{\big(\lambda(T-t)\big)^{x-N_t}}{(\lambda T)^x} \frac{x!}{(x-N_t)!} \mathbb{I}_{\{N_t \le x\}}, \quad \forall\ t \in [0, T).$$

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- ▷ For all $t \in [0, T]$, we get

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Note: p^x have positive probability to jump to zero exactly in correspondence of the jump times of the Poisson process N (condition $\mathbb{P}[\eta^x < \infty, \Delta S_{\eta^x} \neq 0] = 0$ γ-a.e. fails).

Conclusions

- We provide a simple and general condition for preservation of NA1 under filtration enlargement for any fixed semimartingale model.
- We obtain a characterization of NA1 stability under filtration enlargement in a robust context, that is, for all possible semimartingale models.
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Thank you for your attention!